Simple approximate formula for the reflection function of a homogeneous, semi-infinite turbid medium

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A simple, approximate analytical formula is proposed for the reflection function of a semi-infinite, homogeneous particulate layer. It is assumed that the zenith angle of the viewing direction is equal to zero (thus corresponding to the case of nadir observations), whereas the light incidence direction is arbitrary. The formula yields accurate results for incidence–zenith angles less than approximately 85° and can be useful in analyzing satellite nadir observations of optically thick clouds.

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1. INTRODUCTION

The reflection function $R(\theta_0, \theta, \phi)$ is defined as the ratio of the reflected diffused light intensity $I(\theta_0, \theta, \phi)$ for the case of an arbitrary light-scattering, plane-parallel medium to that of a Lambertian ideally white reflector. That is, we have

$$R(\theta_0, \theta, \phi) = \frac{I(\theta_0, \theta, \phi)}{I^*(\theta_0)},$$

where

$$I^*(\theta_0) = F \cos \theta_0$$

is the intensity of light reflected from the ideally white Lambertian reflector, πF is the incident light flux on the area perpendicular to the direction of incidence, θ_0 is the light incidence angle, θ is the observation angle, and ϕ is the relative azimuth between incidence and observation directions. We also will use $\eta = |\cos \theta|$, $\eta_0 = \cos \theta_0$. Clearly, it follows for the Lambertian ideally white reflector that $R \equiv 1$. By definition, this result does not depend on the viewing geometry.

Although nonabsorbing optically thick media are white when looking from the illumination direction, their reflection functions generally differ from 1. The value depends on the viewing geometry, the microstructure and thickness of the medium, and the type of underlying surface.

We will assume in this study that the optical thickness τ is such that the reflection function does not change if we further increase τ (i.e., increase the concentration of particles or geometrical thickness of a turbid medium). Effectively it means that we consider the case of a semi-infinite layer.

The reflection function of a homogeneous, semi-infinite, turbid medium plays an important role both in the radiative transfer theory[1,2] and in planetary optics.[3] In particular, the reflection function of an optically thick scattering layer can be easily calculated if the reflection function of a semi-infinite layer with the same local optical characteristics is known.[1–5]

The primary task of this paper is to obtain an approximate analytical solution for the reflection function of a semi-infinite, plane-parallel layer of cloud water droplets. However, the results obtained can also be applied to other types of turbid media (especially for media with values of the average cosine of the scattering angle that are close to those of cloudy media). The reflection function depends on the angular scattering coefficient of an elementary volume of the turbid medium in question, the viewing and observation geometries, and the probability of photon absorption in the single-scattering event γ. It is defined as the ratio $\sigma_{\mathrm{abs}}/\sigma_{\mathrm{ext}}$, where σ_{abs} and σ_{ext} are absorption and extinction coefficients of the turbid medium. We will assume that $\gamma = 0$, which simplifies the analysis. However, the general approach used here can also be applied, with a slight modification, to the case of weakly absorbing media. Indeed, it is known that[1,2]

$$R_\infty(\eta, \xi, \phi) = R_\infty^0(\eta, \xi, \phi) - yK_0(\eta)K_0(\xi) \tag{1}$$

as $\gamma \to 0$. Here $R_\infty(\eta, \xi, \phi)$ is the reflection function of a semi-infinite, weakly absorbing medium, $R_\infty^0(\eta, \xi, \phi)$ the reflection function of a semi-infinite, nonabsorbing medium, $y = 4k/3(1-g)$, g is the asymmetry parameter, and k is the diffusion exponent of the radiative transfer theory ($k \to [3\gamma(1-g)]^{1/2}$ as $\gamma \to 0$). It follows that $K_0(\eta) = (3/7)(1 + 2\eta)$ at $\eta > 0.2$ with the error less than 2%.[4,5] Equation (1) allows us to find $R_\infty(\eta, \xi, \phi)$ for a weakly absorbing layer if the function $R_\infty^0(\eta, \xi, \phi)$ is known.

2. REFLECTION FUNCTION

The reflection function $R_\infty^0(\eta, \xi, \phi)$ can be represented as the sum of two terms:

$$R_\infty^0(\eta, \xi, \phi) = R_{\mathrm{ss}}(\eta, \xi, \phi) + R_{\mathrm{ms}}(\eta, \xi, \phi). \tag{2}$$

The function $R_{\mathrm{ss}}(\eta, \xi, \phi)$ is the single-scattering contribution given by[6]

$$R_{ss}(\eta,\ \xi,\ \phi) = \frac{p(\vartheta)}{4(\eta + \xi)}, \tag{3}$$

where $p(\vartheta)$ is the phase function and

$$\vartheta = \arccos[-\eta\xi + (1 - \eta^2)^{1/2}(1 - \xi^2)^{1/2}\cos\phi] \tag{4}$$

is the scattering angle.

The function $R_{ms}(\eta,\ \xi,\ \phi)$ gives us the multiple-scattering contribution. Because of the randomizing effect of multiple scattering, its dependence on the relative azimuth angle and the phase function is weak; so we will neglect this dependence. We will use the following approximation for this function:

$$R_{ms}(\eta,\ \xi) = \frac{a + b\eta\xi + c(\eta + \xi)}{4(\eta + \xi)}. \tag{5}$$

Equation (5) was proposed by Sobolev[1] for the description of the azimuthally averaged reflection function of a semi-infinite, nonabsorbing medium. It follows from Eqs. (3)–(5) that

$$R_\infty^0(\eta,\ \xi,\ \phi) = \frac{p(\vartheta)}{4(\eta + \xi)} + \frac{a + b\eta\xi + c(\eta + \xi)}{4(\eta + \xi)}. \tag{6}$$

In order to find the constants a, b, and c, we calculate the function $R_\infty^0(\eta,\ \xi,\ \phi)$ using the exact radiative transfer code.[7] The code is based on the numerical solution of Ambarzumian's nonlinear integral equation.[7] This allows us to find the function

$$D(\eta,\ \xi) = 4(\eta + \xi)R_\infty^0(\eta,\ \xi,\ \phi) - p(\vartheta). \tag{7}$$

Note that it follows from Eqs. (6) and (7) that

$$D(\eta,\ \xi) = a + b\eta\xi + c(\eta + \xi). \tag{8}$$

The function $D(1,\ \xi)$, calculated with Eq. (7) at $\eta = 1$ (nadir observation), is depicted in Fig. 1. Calculations were performed for water droplets with an effective radius of 6 μm at a wavelength of $\lambda = 0.65$ μm. The particle size distribution is given by $f(r) = Nr^6\exp(-9r/r_{\text{eff}})$, where r_{eff} is the effective radius equal to the ratio of the third and second moments of the particle size distribution and N is the normalization constant. The curve in Fig. 1 can be approximated by the following linear relationship,

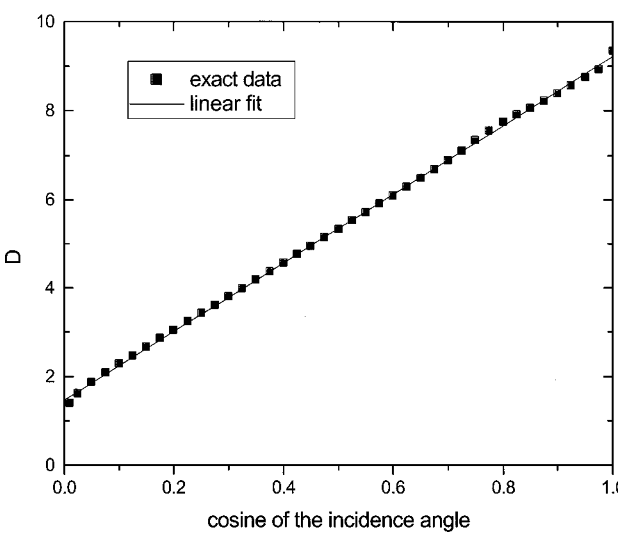

Fig. 1. Dependence $D(1\xi)$ as obtained from Eq. (7) (symbols) and Eq. (9) (solid curve) for $A = 1.468$ and $B = 7.748$.

$$D(1,\ \xi) = A + B\xi, \tag{9}$$

where $A = 1.468$ and $B = 7.748$. On the other hand, it follows from Eq. (8) for $\eta = 1$ that

$$D(1,\ \xi) = a + c + (c + b)\xi. \tag{10}$$

Considering Eqs. (9) and (10), we see that Eq. (10) has three free parameters as compared with the two parameters A and B in Eq. (9). We will assume that $c = 0$. Then it follows that $a \equiv A$ and $b \equiv B$. Equation (6) takes the following final form for $\eta = 1$:

$$R_\infty^0(\theta_0) = \frac{\alpha + \beta\cos\theta_0}{1 + \cos\theta_0} + \frac{p(\pi - \theta_0)}{4(1 + \cos\theta_0)}, \tag{11}$$

where

$$\alpha = a/4 \approx 0.37, \qquad \beta = b/4 \approx 1.94.$$

Equation (11) reduces the calculation of the reflection function of a semi-infinite, nonabsorbing medium at nadir observation to the calculation of the phase function $p(\pi - \theta_0)$. It should be pointed out that the last term in Eq. (11) is small compared with the first term for most incidence angles. This reduces Eq. (11) to

$$R_\infty^0(\theta_0) = \frac{0.37 + 1.94\cos\theta_0}{1 + \cos\theta_0}, \tag{12}$$

which can be used for rapid estimations.

The comparison of calculations of the reflection function for cloudy media with effective droplet radii $r_{\text{eff}} = 6$ and 16 μm based on Eq. (11) and the exact code described in Ref. 7 is presented in Fig. 2(a). One can see that the simple approximation given by Eq. (11) can indeed be applied to calculations of the reflection function of semi-infinite water clouds corresponding to the case of nadir observations. The error is less than 2% at $\theta_0 < 85°$ [see Fig. 2(b)]. For larger solar angles the accuracy of the approximation breaks down as does the assumption of plane-parallel geometry. We note also that the reflection function of semi-infinite clouds at nadir observation is almost independent of the cloud droplet size, which is not surprising given the large size of nonabsorbing cloud droplets relative to the wavelength.

Thus we have managed to parameterize the reflection function of a semi-infinite water cloud at nadir observations with the simple approximate formula of Eq. (11). This allows one to simplify the solution of both the direct and the inverse problems of cloud optics, which are based largely on the asymptotic theory that is valid for optically thick light-scattering layers. For instance, it is known that the reflection function of optically thick clouds in the visible is given by[2,8]

$$R(\theta,\ \theta_0,\ \phi,\ \tau) = R_\infty^0(\theta,\ \theta_0,\ \phi) - tK_0(\cos\theta)K_0(\cos\theta_0), \tag{13}$$

where

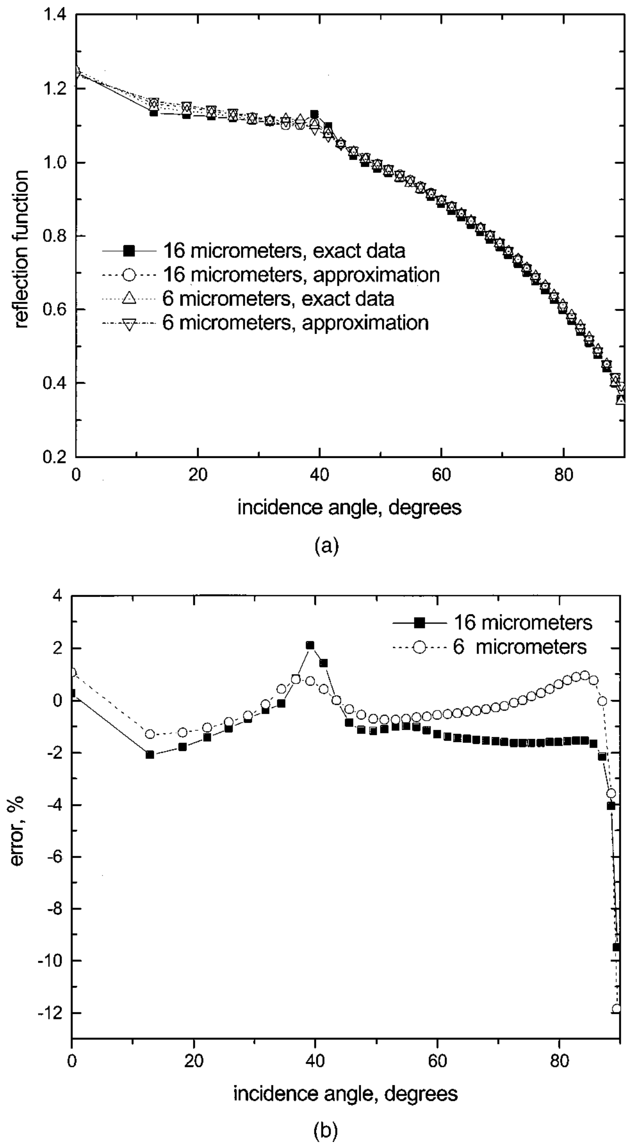

Fig. 2. (a) Dependence of the reflection function $R(1, \xi)$ at nadir observation on the incidence angle θ_0 at $\lambda = 0.65\ \mu$m and effective radii of 6 and 16 μm. (b) Dependence of the error of the approximation of Eq. (11) on the incidence angle at $\lambda = 0.65\ \mu$m and effective radii of 6 and 16 μm.

$$t = \frac{1}{0.75\tau(1 - g) + \alpha}, \tag{14}$$

$\alpha \approx 1.07$, g is the asymmetry parameter, and $R_\infty^0(\theta, \theta_0, \phi)$ is given by Eq. (11). The value of t is the global transmittance, which is equal to $1 - r$ owing to the energy conservation law. Here

$$r = \frac{2}{\pi}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \cos\theta \int_0^{\pi} \mathrm{d}\theta_0 \cos\theta_0 R(\theta_0, \theta, \phi, \tau) \tag{15}$$

is the spherical albedo. We see that integral (15), which is of paramount importance in climate studies, can be easily retrieved from Eq. (13), even if information on the cloud optical thickness and microstructure is not available. That is, it follows that

$$r = 1 - \frac{R_\infty^0(\theta, \theta_0, \phi) - R(\theta, \theta_0, \phi, \tau)}{K_0(\cos\theta)K_0(\cos\theta_0)}, \tag{16}$$

where $R(\theta, \theta_0, \phi, \tau)$ is the measured function.

Note that values of τ and g can be related to the droplet size by means of the following approximate equations[5]:

$$\tau = \frac{3W}{2\rho r_{\mathrm{eff}}}\left(1 + \frac{1.1}{x_{\mathrm{eff}}^{2/3}}\right), \qquad g = 0.88 - \frac{1}{2x_{\mathrm{eff}}^{2/3}}, \tag{17}$$

where $x_{\mathrm{eff}} = 2\pi r_{\mathrm{eff}}/\lambda$, ρ is the specific density of water, and W is the liquid water path. Equations (13), (14), and (17) in combination with Eq. (11) can be applied, e.g., to the liquid water path determination if the value of the effective radius is known (e.g., from measurements of the reflection function in the infrared region of the electromagnetic spectrum).

3. CONCLUSIONS

The main result of this paper is the simple Eq. (11) for the reflection function of a cloudy layer at the nadir observation. It can be applied to the remote sensing of cloud microstructure from space-based platforms. The weak dependence of the reflection function described by Eq. (11) on the cloud droplet sizes means that one can use this function for vertically inhomogeneous clouds without any changes. For rapid estimations one can use the simplified Eq. (12), which depends only on the geometry of the problem.

I believe also that the results and the general parameterization approach can be applied to turbid media other than clouds (especially if they have values of the average scattering angles in the single-scattering event close to those characteristic for cloudy media). Furthermore, the results presented here can be used for the derivation of cloud total reflectance from the reflection function measurements in the visible, which is of importance for climate studies.

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